Comparison Study of Two Inverted Pendulum Models for Balance Recovery

Zhibin Li, Chengxu Zhou, Houman Dallali, Nikos G. Tsagarakis, and Darwin G. Caldwell

Abstract—The inverted pendulum model (IPM) represents better a human-like gait, however, its nonlinearity introduced by the impact during the change of support leg prevents its implementation. The analytic feature of the Linear Inverted Pendulum Model (LIPM) makes it widely applied for the bipedal gait control and balance recovery. We resolve the analytic solution issue for the IPM by using the principle orders in the Taylor series, and further prove the predictive properties of both models. Our theoretical and simulation studies quantitatively compare these two models on the prediction of the target foot placement and allowable swing time. The dynamic simulation and preliminary experiments validated the effectiveness of the IPM based foot placement control.

I. INTRODUCTION

To date, there are more humanoids perform standing balancing [1] [2], but fewer show effective fall prevention under large pushes by taking steps [3] [4] [5]. The stepping strategy for humanoids currently still lacks of common and successful demonstrations. The standing balancing is less challenging if the external disturbance is not large enough to topple the robot. However, as the disturbance increases, the robot must take steps to change the contact polygon to obtain a larger range of wrench for fall prevention.

One notable control approach for the stepping strategy is derived from the zero moment point (ZMP) paradigm [6] by heavily replanning the time-based trajectories. Morisawa et al. modified the foot trajectory according to the detection of disturbance by using the simultaneous planning of the center of mass (COM) and the ZMP [7]. Urata et al. selected the optimal pattern from a number of online re-planned foot and ZMP trajectories and re-generate a new walking pattern for stepping [4]. Their common ground is the use of LIPM to re-plan the walking patterns. Besides, this class of controllers inherently rely on an extra stabilizer that damps out a portion of excessive disturbance at each step, which implies that these controllers might demand more steps than the real systems physically ought to. The other control paradigm applied event-based approach. Pratt et al. proposed Capture Point (CP) predicted by the LIPM to avoid falling [5]. Wight et al. proposed the foot placement estimator as a dynamic balance measure to quantitatively indicate the level of stability, and validated the effectiveness on a planar robot [3]. Only the work of Wight [3] exploited IPM, whereas other methods favored LIPM and the change of the mechanical energy caused by the foot landing impact is not considered.

In fact, IPM is excessively used by the passive dynamic walking (PDW) [9] [10] [11] [12], where the loss of kinetic energy is compensated by the potential energy. It is partially true that the IPM approach is rather challenging for the classical stiff robots, particularly for the human-size humanoids, because the impacts will damage the gear transmissions. However, our study can still be useful for those new platforms empowered by the intrinsic compliance and impedance control technologies [13] [14] with the capability of soft robot-environment interactions.

In the context of one-step push recovery, we study the prediction of position and timing of the foot placement using these two different models, the Linear Inverted Pendulum Model (LIPM) linearized at a constant height in the Cartesian coordinate, and the Inverted Pendulum Model (IPM) linearized at a constant pendulum length in the Spherical coordinate. The authors resolve the nonlinearity issue of IPM by providing analytic solutions based on an acceptable approximation. We also utilize the estimation of future energy state so as to extend the instantaneous capture point [5] to be the predictive variant based on the LIPM; and by our proposed solution for the IPM, the predictive formulas can be derived analytically to extend the original foot placement estimator [3]. We wish to shed some light on the development of applying the PDW principles in powered bipeds, and bridging the gap between the ZMP and PDW communities.

The paper is organized as follows. Section II revisits the original capture point and formulate our extension of its predictive variant. Section III elaborates the principles of using IPM for balance recovery considering the ground impact. Section IV-A quantitatively compares the differences of LIPM and IPM using parameter scan based on the single mass simulation, and Section IV-B presents dynamic simulations of push recovery using the minimum step predicted by the IPM in Robotran simulator. The preliminary experimental validation is demonstrated in Section IV-C. We conclude the study and suggest the future work in Section V.

II. ONE-STEP BALANCE RECOVERY USING LIPM

A. Instantaneous Capture Point

The LIPM has a constraint of constant COM height, defined as as \( z_c \), as shown in Fig. 1 (a). With respect to the instantaneous point support \( x_{cop} \), the orbital energy \( E \) of the LIPM is conserved by its dynamic property [15]

\[
E = \frac{1}{2} x^2 - \frac{g}{2z_c} z^2 \equiv \text{Const.} \tag{1}
\]
To stop right above the point support, the orbital energy shall be zero as shown in Fig. 1 (a). Therefore, setting (1) to zero, the instantaneous support point can be derived

\[ x_{cp} = x + sgn(\dot{x}) \sqrt{\frac{z_c}{g} \dot{x}}. \]  

(2)

In fact, the relative distance \( x_s \) with respect to the COM is of interest, and for the convenience of comparison study, only the \( \dot{x}_0 > 0 \) case is considered and hereafter.

\[ x_{LIPM}^{IPM} = \sqrt{\frac{z_c}{g} \dot{x}}. \]  

(3)

**B. Predictive Capture Point**

The instantaneous Capture Point answers where the foot placement should place given the current COM state of the robot. By predicting the future COM state, the location and time of the foot placement can be computed in advance.

Within the same step, the orbital energy is conserved according to (1) if no external energy is injected,

\[ \frac{\dot{x}_0^2}{2} - \frac{g}{z_c} \left( x_0^2 - \frac{z_c}{g} \dot{x}_0^2 \right) = \frac{\dot{x}_f^2}{2} - \frac{g}{z_c} \left( x_f^2 - \frac{z_c}{g} \dot{x}_f^2 \right). \]  

(4)

Hence, given the COM state feedback \( (x_0, \dot{x}_0) \) at the current moment, and the future target position \( x_f \) where the change of support leg must occur, then the future COM velocity can be predicted by (4) as

\[ \dot{x}_f = sgn(\dot{x}_0) \sqrt{\frac{\dot{x}_0^2 + \frac{g}{z_c} (x_f^2 - x_0^2)}{2}}. \]  

(5)

The transition time from the current state \( (x_0, \dot{x}_0) \) at \( t_0 \) to a future state \( (x_f, \dot{x}_f) \) at \( t_f \) can be obtained [15]

\[ t_s^{IPM} = T_c \ln \left( \frac{x_f + T_c \dot{x}_f}{x_0 + T_c \dot{x}_0} \right). \]  

(6)

Substitute the predicted future COM state \( \dot{x}_f \) computed from (5) into (3), the future Capture Point with respect to the COM can be known in advance, and the allowable swing time is given by (6).

**III. One-Step Balance Recovery Using IPM**

Similarly, the future evolution of the COM state can also be predicted based on the IPM for determining where/when the foot should be placed to avoid falling. The robot as a whole has moment of inertia \( I_c \) around its overall COM. We assume a posture controller could successfully minimize the intrinsic angular momentum close to zero, thus the behavior of the robot approximates that of a point mass. As shown in Fig. 1 (b), the robot is modeled as a massless inverted pendulum with point mass \( m \) and a point foot for the simplicity. The dynamics of intrinsic angular momentum is excluded from our scope for an easier analysis in the following study.

**A. Predictive Foot Placement**

Given an initial velocity, the robot travels forward passively as an inverted pendulum, and has to take a step with unavoidable impact before falling out the friction cone. The momentum around the new point foot (pivot) is conserved, since the collision force exerts no torque around the pivot,

\[ H_0^{n+1} = H_f^n. \]  

(7)

For the angular momentum around the pivot just before the collision, note that only the velocity component perpendicular to the new stance leg \( l_n \omega_f^n \cos(\theta_f^n + \theta_0^{n+1}) \) contributes to the angular momentum,

\[ H_f^n = l_n \omega_f^n = ml_{n+1} \left( l_n \omega_f^n \cos(\theta_f^n + \theta_0^{n+1}) \right). \]  

(8)

The momentum around the pivot after the collision is

\[ H_0^{n+1} = l_{n+1} \omega_0^{n+1} = ml_{n+1} \omega_0^{n+1}. \]  

(9)

Substitute (8) and (9) into (7), we have

\[ \frac{\omega_0^{n+1}}{\omega_f^n} = \frac{l_n}{l_{n+1}} \cos(\theta_f^n + \theta_0^{n+1}). \]  

(10)

Suppose the pendulum length remains constant during a step, therefore the kinetic energy along the pendulum is zero and the kinetic energy of the robot with respect to an instantaneous pivot can be approximated by

\[ \begin{cases} E_{k_f}^n = \frac{1}{2} l_n (\omega_f^n)^2, \\ E_{k_0}^{n+1} = \frac{1}{2} l_{n+1} (\omega_0^{n+1})^2, \end{cases} \]  

(11)

where the inertia around pivot is

\[ \begin{cases} I_f^n = ml_n^2, \\ I_0^{n+1} = ml_{n+1}^2. \end{cases} \]  

(12)

Substitute (10) and (12) into (11), we have

\[ E_{k_0}^{n+1}/E_{k_f}^n = \cos^2(\theta_f^n + \theta_0^{n+1}). \]  

(13)

**TABLE I: Definition of parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_n )</td>
<td>length of pendulum at the ( n ) step</td>
</tr>
<tr>
<td>( E_k^n )</td>
<td>initial kinetic energy at the beginning of the ( n ) step</td>
</tr>
<tr>
<td>( E_{k_f}^n )</td>
<td>final kinetic energy at the end of the ( n ) step</td>
</tr>
<tr>
<td>( \omega_0^n )</td>
<td>initial angular velocity at the beginning of the ( n ) step</td>
</tr>
<tr>
<td>( \omega_f^n )</td>
<td>final angular velocity at the end of the ( n ) step</td>
</tr>
<tr>
<td>( I_n )</td>
<td>inertia tensor with respect to the pivot at the ( n ) step</td>
</tr>
<tr>
<td>( H_0^n )</td>
<td>angular momentum at the beginning of the ( n ) step</td>
</tr>
<tr>
<td>( H_f^n )</td>
<td>angular momentum at the end of the ( n ) step</td>
</tr>
</tbody>
</table>
It implies that the percentage of preserved kinetic energy due to the change of the support leg only depends on the inter-leg angle $\theta_f^n + \theta_0^{n+1}$, regardless of the length of legs.

Given an initial angle of landing foot $\theta_0^{n+1}$ at the $n+1$ step, the negative work done by the gravity from $\theta_0^{n+1}$ to the upright angle shall be

$$W = mgl_{n+1} \left( \cos(\theta_0^{n+1}) - 1 \right). \quad (14)$$

If the COM rests right above the foot, the negative work of gravity cancels the kinetic energy $E_{k_0}^{n+1}$ at $n+1$ step,

$$E_{k_0}^{n+1} + W = 0. \quad (15)$$

Recall (13), we have the one step balance recovery function $F$ determined by the inverted pendulum model

$$E_{k_f}^{n} \cos^2(\theta_f^n + \theta_0^{n+1}) + mgl_{n} \cos \theta_f^n \frac{\cos(\theta_0^{n+1}) - 1}{\cos \theta_0^{n+1}} = 0. \quad (16)$$

The geometric relation of stance legs on the flat terrain is

$$l_{n+1} = l_n \cos \theta_f^n / \cos \theta_0^{n+1}. \quad (17)$$

Substitute (17) into (16), yields

$$E_{k_f}^{n} \cos^2(\theta_f^n + \theta_0^{n+1}) + mgl_{n} \cos \theta_f^n \frac{\cos(\theta_0^{n+1}) - 1}{\cos \theta_0^{n+1}} = 0. \quad (18)$$

Hence, the one step balance recovery function $F$ is determined by the kinetic energy, length and angle of stance leg prior to the collision of the new support foot.

Let (18) be divided by $m$ and substitute the term $E_{k_f}^{n}$ according to (11) and (12), yields

$$\frac{1}{2}l_n^2 \omega_0^n \cos^2(\theta_f^n + \theta_0^{n+1}) + gl_{n} \cos \theta_f^n \frac{\cos(\theta_0^{n+1}) - 1}{\cos \theta_0^{n+1}} = 0. \quad (19)$$

Define intermediate variables

$$\begin{cases} 
  a = \frac{1}{2}l_n^2 \omega_0^n, \\
  b = gl_{n} \cos \theta_f^n, \\
  c = \theta_0^n,
\end{cases} \quad (20)$$

and substitute into (19), we obtain a neat expression of $F$

$$F = a \cos^2(c + \theta_0^{n+1}) + b \left( 1 - \frac{1}{\cos \theta_0^{n+1}} \right) = 0. \quad (21)$$

Given an arbitrary $\theta_0^{n+1}$, $F$ in (21) can quantify whether or not the robot would travel over the new stance foot. $F < 0$ means the robot will return before potential apex; $F = 0$ indicates a rest at potential apex; $F > 0$ suggests falling. The nonlinear equation (21) has no analytic solution. However, since the foot landing angle must be restricted within the friction cone, the range of $\theta_0^{n+1}$ is limited. Therefore, an approximation can be derived with small resulted error by using the principle components of the Taylor series of (21).

Defined $F_{(2)}$ as the 2nd order Taylor-polynomial of (21),

$$F_{(2)} = a \cos^2(c) - 2a \cos(c) \sin(c)\theta_0^{n+1} + (a \sin^2(c) - a \cos^2(c) - 0.5b)(\theta_0^{n+1})^2. \quad (22)$$

For quantitatively examining the error caused by the approximate solution prior to rigorous study, a numerical iteration algorithm is developed to find the exact solution of (21). The limit of numerical error is set as $|F| \leq 1.0 \times 10^{-6}$.

We validate the accuracy of the approximate solution by parameter scan. The initial conditions are $\theta_0 = 0^\circ$ and $l = 1m$. The horizontal velocity $v_0 (v_0 = \omega d)$ varies from 0 to 2 m/s, and the final angle of stance leg $\theta_f$ shifts from 0 to 40$^\circ$. Thus

$$x_{s_F}^{p_M} = l_n \cos \theta_f^n \tan \theta_0^{n+1}. \quad (24)$$

Perform the parameter scan of $v_0$ and $\theta_f$, and substitute (25) into (21) and (23) respectively for the exact and the approximate solutions. Fig. 2 shows these two solutions are almost identical. Given a wide range of parametric variations, the errors of step distance are no more than 2cm as shown in Fig. 3(a). Therefore, it is convincing that the approximated solution (23) and (24) can be used for real implementation. The number of iterations shown in Fig. 3(b) suggests that computational cost of numerical solution is dependent on
Fig. 4: Parameter scan of one-step balance recovery using LIPM (red) and IPM (green): (a) and (b) show the comparison of step distance predicted by two models; (c) and (d) are the step distance prediction by LIPM and IPM respectively.

Fig. 5: Step difference and swing time predicted by LIPM and IPM

Fig. 6: Comparison of foot placement using (a) LIPM with step size 0.57m and (b) IPM with step size 0.32m.

the initial conditions. The number of iterations is particularly high at low/medium velocity and small COM displacement, thus the numerical solution wouldn’t warrant a safe and fixed computational time for real-time implementation.

B. Predictive Step Time

Similarly to the LIPM, the IPM has the constant pendulum length around the pivot, where the linearization in the spherical coordinate will result in analytic solutions for the predictive foot placement in a similar analogy.

The conservation of the mechanical energy yields

$$\begin{align*}
E_{k0} &= \frac{1}{2} ml^2 \dot{\theta}_0^2, \\
E_{k_f} &= E_{k0} + mgl(\cos \theta_0 - \cos \theta_f).
\end{align*}$$

(26)

Given the COM state feedback ($\theta_0$, $\dot{\theta}_0$) at the current moment, and the target angular position $\theta_f$ of the future COM, then the corresponding future COM velocity $\dot{\theta}_f$ is

$$\dot{\theta}_f = sgn(\dot{\theta}_0) \sqrt{\dot{\theta}_0^2 + 2 \frac{g}{l}(\cos \theta_0 - \cos \theta_f)}. \hspace{1cm} (27)$$

Given $\dot{\theta}_f$ from (27), the transition time from the current state ($\theta_0$, $\dot{\theta}_0$) to a predicted future state ($\theta_f$, $\dot{\theta}_f$) can be approximated by a similar analytic formula to (6)

$$\theta^{PM}_s = T_c \ln \left( \frac{\theta_f + T_c \dot{\theta}_f}{\theta_0 + T_c \dot{\theta}_0} \right). \hspace{1cm} (28)$$

Note that the angular position formulated in (23) already inherits the foot placement prediction given by the future position $\theta_f$, while (28) analytically provide the approximated time that allows the foot to swing.

IV. SIMULATION STUDY OF LIPM AND IPM

A. Simulation of Single Mass Dynamics

The simulations of the theoretical analysis focus on the point mass dynamics, omit the dynamical effect of the swing leg, and consider the instantaneous and plastic collisions. Numerical simulations based on the dynamics of the point mass allow the validation of the proposed formulations, and further systematically evaluate the differences between models. The analytical solutions (23) and (24) of the approximate function (22) are used for the parameter scan of IPM.

To compare in an unbiased manner, the initial conditions are: ($0$, $l_0$) and ($x = v_0$, $z = 0$) for LIPM, ($0$, $l_0 \cos \theta_0$) and ($x = v_0$, $z = 0$) for IPM. The initial position is configured such that the COM position is the same in both models when the foot landing event occurs. $l_0$ is the nominal length of the leg. Since the LIPM imposes the constant height constraint $z_c$, setting $z_c = l_0 \cos \theta_0$ avoids the knee singularity for the articulated leg. The initial height of IPM is $l_0$ because the compass like stepping permits a straight leg configuration.

Fig. 4 shows the difference in step distance with respect to the COM given a range of the initial velocity and the final angle of the stance leg. The step distance increases almost linearly for the LIPM but not for the IPM. In general, the IPM suggests a smaller step since it considers the energy loss from the impact. The elapsed time from the initial condition to the new step is the swing time that allows the foot to place at the new foothold. It can be seen that despite the step difference predicted by two models is quite obvious (Fig. 5(a)), however, there is no big difference in swing time as Fig. 5 (b) suggests.

Fig. 6 shows the time elapsed figure of one-step balance recovery scenario. In both cases, the switch of the support leg occurs at the same COM position before exiting the friction cone ($\mu=0.8$), as highlighted by the red line which is $30^\circ$ from the vertical line. The initial velocity is $1 m/s$ at the vertical position and the step distance is $0.57 m$ and $0.32 m$ respectively based on the LIPM and IPM dynamics. The step
length suggested by the IPM is much shorter and realistic. Our study implies that the IPM requires less actuation power because it demands a smaller step within approximately the same swing time. Moreover, IPM will seldom cause singularities since the step distance is more likely to be within the leg’s workspace. Contrarily, the singularity is an issue for the LIPM in the case of high velocity and large COM displacement which requests a larger step. To avoid the singularity, the LIPM based control compromises with unnecessary multiple steps to stop under the same push.

B. Simulation of Multi-body Dynamics

The predictive foot placement control based on the IPM was validated by a seven-segment model in the multi-body simulation in Robotran [16]. The mass property was set close to that of the COMAN humanoid robot [13] without arms. The body, thigh, calf and foot had the mass of 12.0kg, 4.0kg, 3.0kg and 1.0kg, and the length of 0.42m, 0.22m, 0.2m, 0.08m. Each segment had its COM at the geometric center. The inertia was configured assuming homogeneous distribution of mass over the cylinder and ellipse volumes.

The initial velocity of the pelvis was used as the push disturbance to examine the foot placement control. All the joints were position controlled. During the single support phase, the support leg was steered at the velocity reference computed in proportional to the torso orientation error in order to keep torso upright. The steering of the swing leg was governed by the PD law using the error between the measured touch down angle (pointing from the COM of the robot to the center of the swing foot) and the foot placement angle predicted by (23). The virtual leg of the swing leg, pointing from hip to ankle, was actively controlled to be compliant for the impact absorption. The stiffness of the virtual leg was 8000 N/m and the viscous constant was 10 Ns/m. The leg compliance was achieved by the admittance control which was adopted from the 1-DOF rotational formulation in [17]. The compliance control modulated the referential leg length based on the contact force sensed in the foot.

After the swing foot touched the ground, both legs stopped spinning and kept the same leg angle with respect to the body. The new support leg remained compliant to buffer the landing motion, and thus was shortened later by the body weight as shown in Fig. 7. The previous support leg started to modulate the leg length according to the length of the new support leg based on the geometric relation of the lower body for keeping the pelvis level to the ground.

Fig. 8 shows the comparison of the step size by the instantaneous capture point, and the predictive foot placement by the LIPM and IPM respectively. The initial velocity disturbances applied at the pelvis were 0.2 m/s, 0.4 m/s, and 0.6 m/s. Both the extended predictive variants based on LIPM and IPM computed the foot placements in advance, and varied little until the touch down events occurred. But the instantaneous capture point was evolving during falling. Since the tracking control of foot placement inevitably has latency, the real foot position will always lag behind the instantaneous capture point or the original foot placement, leading to tedious tuning of safety margins. On the contrary, our proposed prediction provides the time margin about 0.2s for the tracking controller to position the foot in time.

C. Preliminary Experiments

The same control scheme was also validated on the half body COMAN robot. The feet of COMAN were modified
TABLE II: Comparison of IPM and LIPM

<table>
<thead>
<tr>
<th>Property</th>
<th>IPM</th>
<th>LIPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>Non-linear</td>
<td>Linear</td>
</tr>
<tr>
<td>Solution</td>
<td>Approximate analytic</td>
<td>Analytic</td>
</tr>
<tr>
<td>Step transition</td>
<td>Included</td>
<td>Not included</td>
</tr>
<tr>
<td>Step distance</td>
<td>Small</td>
<td>Large</td>
</tr>
<tr>
<td>Leg Reachability</td>
<td>Good</td>
<td>Poor</td>
</tr>
<tr>
<td>Swing time</td>
<td>Equal</td>
<td>Equal</td>
</tr>
<tr>
<td>Actuation</td>
<td>Low knee torque</td>
<td>High knee torque</td>
</tr>
<tr>
<td>Disturbance rejection</td>
<td>Higher</td>
<td>Lower</td>
</tr>
</tbody>
</table>

with human-like shapes for wearing shoes, since the control method supported the under-actuation. Fig. 9 shows the snapshots from the preliminary experiment with 0.16s time interval. The control was activated only when the ground projection of the COM moved out of the support feet, starting from the 2nd snapshot in Fig. 9. The real-time performance can be seen in the accompanied video. The authors regret that the experimental data were lost due to the computer crash thus unfortunately cannot be included in this manuscript.

V. REMARKS AND FUTURE WORK

Our comparison study of LIPM and IPM for the balance recovery of bipeds contributes as follows. The theoretical work derives the predictive capture point and foot placement from LIPM and IPM respectively, in addition to their instantaneous versions. An approximate equation for obtaining the analytic solution for the nonlinear IPM is proposed by using the principle orders of the Taylor expansion. The single mass simulation shows systematic evaluations of LIPM and IPM in terms of step size and time.

The features of these two models are summarized in Table II. In the LIPM, all the excessive energy is purely absorbed by the negative work of actuators, whereas, the change of support leg in the IPM redirects the velocity and significantly dissipates the kinetic energy which maximizes the disturbance rejection. Also, the IPM allows a more straight knee configuration thus demands less knee torque during standing and foot landing. The swing time is almost the same for both LIPM and IPM, since the IPM requires a smaller step size, the required step velocity is smaller thus less probability of saturating the actuator limit of the hip.

Our reported dynamic simulation and experimental study were carried out in the lateral scenario that simplifies the balance coordination with the sagittal plane. The future work might consider the push recovery extension in the three dimensional case and a similar implementation of LIPM as a consolidated benchmark.

REFERENCES


